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# Note

# On Convex Approximation by Quadratic Splines

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In a recent paper by Hu it is proved that for any convex function $f$ there is a $C^1$ convex quadratic spline $s$ with $n$ knots that approximates $f$ at the rate of $\omega_3(f, n^{-1})$. The knots of the spline are basically equally spaced. In this paper we give a simple construction of such a spline with equally spaced knots. © 1996 Academic Press, Inc.

In a recent paper [1] Hu proves that for any convex function $f$ there is a $C^1$ convex quadratic spline $s$ with $n$ knots that approximates $f$ at the rate of $\omega_3(f, n^{-1})$. The knots of the spline constructed in [1] are "basically equally spaced". We give here a simple construction of such a spline with equally spaced knots.

THEOREM. *Let $f \in C[0, 1]$ be convex and let n be natural. There is a $C^1$ convex quadratic spline s with knots at $i/n$, such that*

$$\|f - s\| \leqslant c\omega_3(f, n^{-1}), \tag{1}$$

*where c is an absolute constant, $\|\cdot\|$ and $\omega_3$ are the uniform norm and the third uniform modulus of smoothness in $[0, 1]$ respectively.*

*Proof.* Set $h = n^{-1}$, $m = [n/3] - 1$ and $x_i = 3ih$ for $i = 0, 1, ..., m+1$. For $i = 1, 2, ..., m$ let $P_i(x) = A_i x^2 + B_i x + C_i$ be the parabola interpolating $f$ at the points $x_{i-1}$, $x_i$, $x_{i+1}$, i.e.,

$$P_i(x_k) = f_k := f(x_k) \qquad \text{for} \quad k = i-1,\ i,\ i+1. \tag{2}$$

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From (2) and the convexity of $f$ we have

$$A_i = \tfrac{1}{18} n^2 (f_{i+1} - 2f_i + f_{i-1}), \qquad A_i \geqslant 0. \tag{3}$$

Set $g(x) = P_1(x)$ for $x \in [0, x_1]$, $g(x) = P_m(x)$ for $x \in [x_m, 1]$. For $i = 2, 3, ..., m$ set

$$g(x) = \begin{cases} P_i(x) & \text{if } \ A_i \leqslant A_{i-1}; \\ P_{i-1}(x) & \text{if } \ A_{i-1} \leqslant A_i \end{cases} \qquad \text{for} \quad x \in [x_{i-1}, x_i]. \tag{4}$$

The function $g$ is a continuous quadratic spline interpolating $f$ at the knots $\{x_i\}$. For the distance between $f$ and $g$ we have from Whitney's theorem [3]

$$\|f - g\| \leqslant c_1 \omega_3(f, n^{-1}). \tag{5}$$

An important observation is that $g$ is convex (see Remark 1). The only property, which $g$ lacks to satisfy the theorem, is the discontinuity of $g'$ at the knots.

The function $ax_+^2 + bx_+$ can be smoothed to $C^1$ spline with knots $-1$, 0, 1 by

$$\sigma(a, b; x) = ax_+^2 + bx_+ + \tfrac{1}{4} b (1 - |x|)_+^2. \tag{6}$$

In order to apply (6) we calculate the differences between the parabolas interpolating $f$ at $x_i$. From (2) we obtain

$$\begin{aligned} P_i(x) - P_{i-1}(x) &= (A_i - A_{i-1})(x - x_i)(x - x_{i-1}) \\ &= h^2 (A_i - A_{i-1})\, n(x - x_i)(n(x - x_i) + 3), \end{aligned} \tag{7}$$

$$P_{i+1}(x) - P_i(x) = h^2 (A_{i+1} - A_i)\, n(x - x_i)(n(x - x_i) - 3). \tag{8}$$

Adding (7) and (8) we get

$$\begin{aligned} &P_{i+1}(x) - P_{i-1}(x) \\ &\quad = h^2 (A_{i+1} - A_{i-1})\, n^2 (x - x_i)^2 + 3h^2 (2A_i - A_{i+1} - A_{i-1})\, n(x - x_i). \end{aligned} \tag{9}$$

Now we change $g$ on the intervals $[x_i - h, \ x_i + h]$ to a smoother function $s$ in order to get a convex approximant to $f$. Define $s(x) = g(x)$ for $x \notin \bigcup_{i=1}^m [x_i - h, \ x_i + h]$. For every $[x_i - h, \ x_i + h]$, $i = 1, 2, ..., m$, we define $s$ as follows ($A_0 = A_1$, $A_{m+1} = A_m$):

*Case* 1. $A_{i-1} \geqslant A_i \leqslant A_{i+1}$. We set $s(x) = P_i(x) (= g(x))$.

*Case* 2. $A_{i-1} < A_i \leqslant A_{i+1}$. Then $g = P_{i-1}$ in $[x_{i-1}, x_i]$ and $g = P_i$ in $[x_i, x_{i+1}]$. Having in mind (6) and (7) we set

$$s(x) = P_{i-1}(x) + h^2 \sigma(A_i - A_{i-1}, 3A_i - 3A_{i-1}; n(x - x_i)). \tag{10}$$

From (6) and (10) we get $s(x)-g(x)=\frac{3}{4}(A_i-A_{i-1})\,h^2(1-n\,|x-x_i|)^2$ and hence

$$\begin{aligned}0 &\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(A_i-A_{i-1})\\ &=\tfrac{1}{24}(f_{i+1}-3f_i+3f_{i-1}-f_{i-2})\leqslant \tfrac{1}{24}\omega_3(f,3n^{-1})\leqslant \tfrac{9}{8}\omega_3(f,n^{-1}).\end{aligned} \tag{11}$$

*Case* 3. $A_{i-1}\geqslant A_i>A_{i+1}$. Having in mind (6) and (8) we set

$$s(x)=P_i(x)+h^2\sigma(-A_i+A_{i+1},3A_i-3A_{i+1};n(x-x_i)). \tag{12}$$

From (6) and (12) we get

$$0\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(A_i-A_{i+1})\leqslant \tfrac{9}{8}\omega_3(f,n^{-1}). \tag{13}$$

*Case* 4. $A_{i-1}<A_i>A_{i+1}$. Having in mind (6) and (9) we set

$$s(x)=P_{i-1}(x)+h^2\sigma(A_{i+1}-A_{i-1},3(2A_i-A_{i+1}-A_{i-1});n(x-x_i)). \tag{14}$$

From (6) and (14)

$$\begin{aligned}0 &\leqslant s(x)-g(x)\leqslant \tfrac{3}{4}h^2(2A_i-A_{i+1}-A_{i-1})\\ &=\tfrac{1}{24}(-f_{i+2}+4f_{i+1}-6f_i+4f_{i-1}-f_{i-2})\leqslant \tfrac{1}{12}\omega_3(f,3n^{-1})\leqslant \tfrac{9}{4}\omega_3(f,n^{-1}).\end{aligned} \tag{15}$$

From (11), (13) and (15) we get

$$\|s-g\|\leqslant \tfrac{9}{4}\omega_3(f,n^{-1}),$$

which together with (5) implies (1).

The convexity of $s$ will follow from the non-negativity of the leading coefficients of its parabolic components. Outside $\bigcup_{i=1}^{m}[x_i-h,\ x_i+h]$ these coefficients are non-negative in view of (3). In $[x_i-h,x_i]$ and $[x_i,x_{i+h}]$, $i=1,2,\ldots,m$, the coefficients are (see (6), (10), (12) and (14))

$$\begin{aligned}&A_{i-1}+\tfrac{3}{4}(A_i-A_{i-1})\geqslant 0\\ &\qquad\text{and } A_i+\tfrac{3}{4}(A_i-A_{i-1})\geqslant 0 \text{ in Case 2;}\\ &A_i+\tfrac{3}{4}(A_i-A_{i+1})\geqslant 0\\ &\qquad\text{and } A_{i+1}+\tfrac{3}{4}(A_i-A_{i+1})\geqslant 0 \text{ in Case 3;}\\ &A_{i-1}+\tfrac{3}{4}(2A_i-A_{i+1}-A_{i-1})\geqslant 0\\ &\qquad\text{and } A_{i+1}+\tfrac{3}{4}(2A_i-A_{i+1}-A_{i-1})\geqslant 0 \text{ in Case 4;}\end{aligned} \tag{16}$$

respectively. This completes the proof. ∎

*Remark* 1. A basic step in the proof is the simple (although non-linear) construction in (4) of the *convex* continuous quadratic spline $g$. The convexity follows from the convexity of the parabolic components and the positive jumps of $g'$ at $x_i$, which are $3A_i - 3A_{i-1}$, $3A_i - 3A_{i+1}$ and $3(2A_i - A_{i-1} - A_{i-1})$ in cases 2, 3 and 4 respectively.

*Remark* 2. Hu, Leviatan and Yu show in [2, Theorem 3] that for any convex function $f$ there is a $C^2$ convex *cubic* spline $S$ with $O(n)$ equally spaced knots, such that

$$\|f - S\| \leqslant c\omega_3(f, n^{-1}), \qquad \|S'''\| \leqslant cn^3\omega_3(f, n^{-1}). \tag{17}$$

One can easily construct such a cubic spline by smooting the spline

$$s(x) = \alpha + \beta x + \sum_{j=0}^{n-1} \gamma_j (x - jh)_+^2 \tag{18}$$

from Theorem 1, where coefficients $\gamma_j$ satisfy (see (16) and (3))

$$|\gamma_j| \leqslant \tfrac{5}{2} \max_i |A_i - A_{i-1}| \leqslant \tfrac{15}{4} n^2 \omega_3(f, n^{-1}). \tag{19}$$

Simply set

$$S(x) = s(x) + \sum_{j=1}^{n-1} \left(\frac{1}{4n}\right)^2 |\gamma_j|\, \eta(4n(x - jh) \operatorname{sign} \gamma_j),$$

$$\eta(x) = \theta(x) - x_+^2, \qquad \theta(x) := \frac{1}{9}(x+1)_+^3 + \frac{1}{6} x_+^3 - \frac{1}{3}(x-1)_+^3 + \frac{1}{18}(x-2)_+^3.$$

Note that $S$ is a cubic spline with $4n$ equally spaced knots, $S^{(k)}(jh - h/2) = s^{(k)}(jh - h/2)$, $k = 0, 1, 2$, $j = 1, 2, ..., n$. (17) follows from (1) and (19). Finally, the convexity of $S$ can be verify as follows. In $x \in [jh, jh + h]$, $j = 0, 1, ..., n-1$, write $s$ as the parabola $Q_j$ and note its convexity in view of the convexity of $s$. From (18) the difference between two consecutive parabolas is $Q_j(x) - Q_{j-1}(x) = \gamma_j (x - jh)_+^2$. Then in $[jh - h/2, jh + h/2]$, $j = 1, 2, ..., n-1$, we have

$$S(x) = Q_{j-1}(x) + \left(\frac{1}{4n}\right)^2 \gamma_j \theta(4n(x - jh)) \qquad \text{if} \quad \gamma_j \geqslant 0;$$

$$S(x) = Q_j(x) + \left(\frac{1}{4n}\right)^2 |\gamma_j|\, \theta(-4n(x - jh)) \qquad \text{if} \quad \gamma_j < 0.$$

This representation implies the convexity of $S$ because $\theta$ and $Q_j$ are convex.

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## References

1. Y. K. Hu, Convex approximation by quadratic splines, *J. Approx. Theory* **74** (1993), 69–82.
2. Y. K. Hu, D. Leviatan and X. M. Yu, Convex polynomial and spline approximation in $C[-1, 1]$, *Constr. Approx.* **10** (1994), 31–64.
3. H. Whitney, On functions with bounded $n$th difference, *J. Math. Pure Appl.* **36** (1957), 67–95.

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